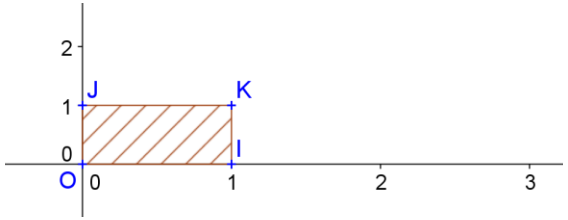
Chapter 7 : Integral Calculus.

In this chapter, we consider an orthogonal frame of the plane.

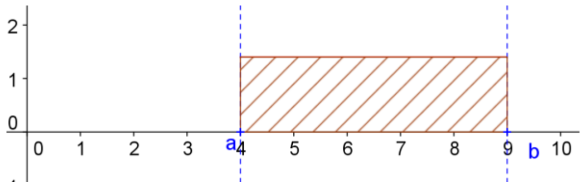
We call area unit (a.u.) the unit of the rectangle $OIKJ$.



1- Notion of integral :

1.1. Constant function :

Let f be a function defined on $]a;b[$ by $f(x) = c$, $c > 0$. We call **integral of f on $]a;b[$** the area of the rectangle defined by the graph of f, the $x - axis$, the straight lines with equation $x = a$ and $x = b$. Its value is $c(b-a)$ a.u.

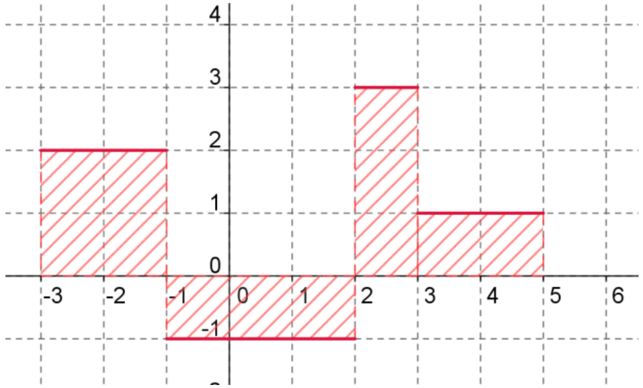


When $c < 0$, we agree to define the integral of f on $]a;b[$ the opposite of the area below : $c(b-a)$, it is an algebraic area, negative in that case.

1.2. Stair Function :

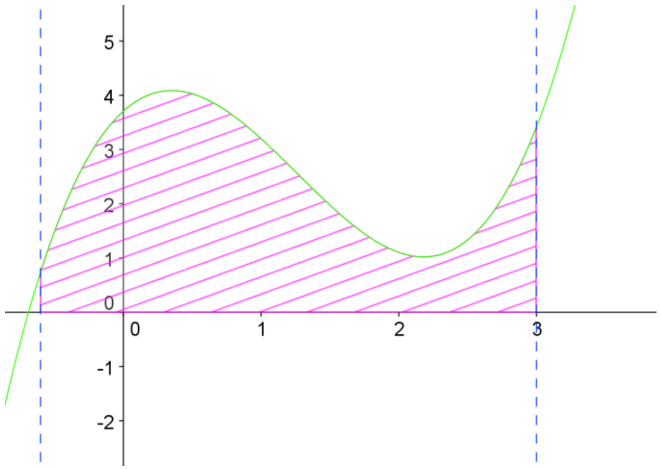
For a stair function (constant function by bits), the integral of f on $]a;b[$ is the algebraic sum of the colored rectangles, counted positively if they're above the $x - axis$, negatively if they're below the $x - axis$.

We denote it $\int_a^b f(x)dx$, or $\int_a^b f(u)du$, or $\int_a^b f(t)dt$…(notation introduced by Leibniz, in the XVII th century).



1.3. Positive continuous function:

Definition : the integral from a to b of f, denoted $\int_a^b \boldsymbol{f(x)dx}$, is the area of the domain bounded by the graph of f, the $x - axis$, the straight lines with equation $x = a$ and $x = b$, in area unit. We also talk about the **area under the curve** from $x = a$ to $x = b$.



Rqe : $\int_a^a f(x)dx = 0$, because the area is the one of a segment line.

2- First properties :

2.1. Extended definition :

- In the case of a negative continuous function, if $a < b$, we write : $\int_a^b f(x)dx = \int_a^b \big(-f(x)\big)dx$, where $-f$ is a positive function.
- For a positive continuous function, and if $a \geq b$, then :

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

2.2. Chasles Law :

f is a continuous function on an interval I. For all real numbers a, b, c in I, we have :

$$\boxed{\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx}$$

Csq : If $f$is odd $\int_{-a}^a f(x)dx = 0$, and if it's even, $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.

2.3. Linearity :

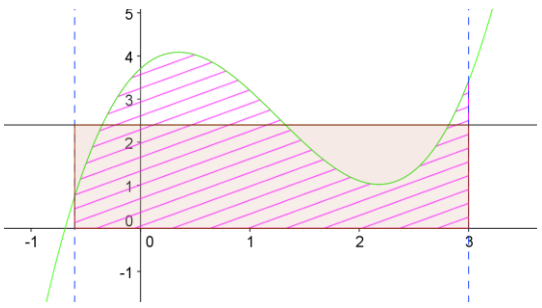
$f$and g both continuous functions on I, and λ a real number. Qfor all real numbers a and b in I we have :

$$\boxed{\int_a^b \big(f(x) + \lambda g(x)\big)\,dx = \int_a^b f(x)dx + \lambda\int_a^b g(x)dx}$$

2.4. Central point :

f is a continuous function on I. For all real numbers a, b in I, **the central point of f on** $[\boldsymbol{a}; \boldsymbol{b}]$ is :

$$\boxed{\mu = \frac{1}{b-a}\int_a^b f(x)dx}$$



It is actually the height of the rectangle with base $(b-a)$ which area is equal to the area under the curve of f on $[a; b]$.

Theorem : For all real numbers a, b in I, we can find a real number $c \in]a; b[$ such that $\boxed{\int_a^b f(t)dt = (b-a)f(c)}$.

2.5. Inequalities :

If for all real number x of $[a; b]$ we have $f(x) > g(x)$, then $\int_a^b f(x)dx > \int_a^b g(x)dx$.

Csq : If $f$est bounded on $[a; b]$, $ie\ m \leq f(x) \leq M$, then we have :

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

3- Antiderivative :

3.1. Definition :

f is a function defined on an interval I. An **antiderivative** of f on I is a differentiable function F such that for all real number x of I : $\boxed{F'(x) = f(x)}$.

Theorem : If F and G are both antiderivative of f, then we can find a real number k such that : $F(x) = G(x) + k$, for all $x \in I$.

Csq : Given a pair of real numbers $(x_0; y_0)$, there is only one antiderivative of f such that : $F(x_0) = y_0$ (boundary condition (initial condition if $t = 0$), in physics most of the time)

Ex : Find the antiderivative of the following functions satisfying the boundary condition :

(a) $f(x) = x$, and $F(0) = 2$
(b) $g(x) = 4x + 1$ and $G(1) = -3$
(c) $h(x) = \frac{1}{x^2}$ and $H(-1) = 0$

3.2. Antiderivative of a continuous function :

> **Theorem :** f is a continuous function defined over I, $a \in I$.
>
> Then the function defined by
>
> $$F(x) = \int_a^x f(t)dt$$
>
> Is the **unique antiderivative** of f that is equal to 0 for $x = a$.

Proof !

- F is differentiable : Indeed, for $x_0 \in I$ and $h \neq 0$ we have :

$$\frac{F(x_0+h)-F(x_0)}{h} = \frac{1}{h}\left(\int_a^{x_0+h} f(t)dt - \int_a^{x_0} f(t)dt\right) = \frac{1}{h}\int_{x_0}^{x_0+h} f(t)dt$$

We have seen that there is a real number c comprised between x_0 and $x_0 + h$ such that $hf(c) = \int_{x_0}^{x_0+h} f(t)dt$.

Then : $\frac{F(x_0+h)-F(x_0)}{h} = \frac{1}{h}\, h\, f(c) = f(c)$.

But, as f is a continuous function, when h tends to 0, c tends to x_0, then $f(c)$ tends to $f(x_0)$.

Therefore : $\lim\limits_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$ which is a real number. Then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

- F is null at a : $F(a) = \int_a^a f(t)dt = 0$.

- F is unique : Let's imagine that G is another antiderivative of f null at a. Then we have $G(x) = F(x) + k$ for a rel number k and for all x in I.
  But $G(a) = 0 = F(a) + k = k$.
  Then $k = 0$ and $F = G$. QED.

Csq : The function $x \mapsto \ln x$ is the antiderivative of $\frac{1}{x}$ that is null at 1 on $]0\,;+\infty[$:

$$\ln x = \int_1^x \frac{1}{t}dt$$

4- Evaluating antiderivative :

4.1. Usual functions :

Function f	Antiderivative F	Intervall $I = \cdots$
$a\ (constante)$	$ax + C$	$\mathbb{R}$
$x^n\ (n \in \mathbb{Z} - \{-1\})$	$\frac{x^{n+1}}{n+1} + C$	$\mathbb{R}$ $if\ n > 0$ $]-\infty\,;0[$ or $]0\,;+\infty[$ $if\ n < -1$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + C$	$]0\,;+\infty[$
$\frac{1}{x}$	$\ln x + C$	$]0\,;+\infty[$
e^x	$e^x + C$	$\mathbb{R}$
$\cos x$	$\sin x + C$	$\mathbb{R}$
$\sin x$	$-\cos x + C$	$\mathbb{R}$
$1 + \tan^2 x = \frac{1}{\cos^2 x}$	$\tan x + C$	$\left]-\frac{\pi}{2} + k\pi\,;\frac{\pi}{2} + k\pi\right[, k \in \mathbb{Z}$

4.2. More formulas :

The operations on differentiable functions as well as the definition of an antiderivative lead to the following results :

- If F and G are antiderivatives of f and g, then $F + G$ is an antiderivative of $f + g$.

- If F is an antiderivative of f and λ is a real number, then λF is an antiderivative of λf.

Function f	Antiderivative F	Remarks
$u'u^n \ (n \in \mathbb{Z} - \{-1\})$	$\frac{u^{n+1}}{n+1}$	If $n < -1$, only for u never null on I.
$\frac{u'}{\sqrt{u}}$	$2\sqrt{u}$	$u > 0$
$\frac{u'}{u}$	$\ln u$ $\ln(-u)$	$u > 0$ $u < 0$
$u'e^u$	e^u	
$x \mapsto u(ax+b), a \neq 0$	$x \mapsto \frac{1}{a}U(ax+b),$	U antiderivative of u.

Example : An antiderivative of $f(x) = x\cos(x^2)$ is $F(x) = \frac{1}{2}\sin(x^2)$.

5- Integral calculus :

5.1. Link between integral and antiderivative :

Fundamental theorem of calculus : f is a continuous function on an interval I, F an antiderivative of f on I, a and b two real numbers belonging to I. Then we have :

$$\int_a^b f(x)dx = F(b) - F(a), \text{often denoted } [F(x)]_a^b$$

Ex : An antiderivative of $f(x) = \cos x$ is $f(x) = \sin x$, then :

$\int_0^{\pi} \cos t\, dt = [\sin t]_0^{\pi} = \sin \pi - \sin 0 = 0.$

Ex : Evaluate the following integrals : $\int_{-1}^{2} x^2\, dx$, $\int_0^2 (2x+3)^3\, dx$, $\int_0^{\frac{\pi}{2}} \sin x \cos^2 x\, dx$, $\int_0^3 \frac{1}{1+2x} dx$, $\int_{-3}^{5} \frac{2}{1+2x} dx$,

5.2. Integration by parts :

Theorem : u and v two derivative functions on an interval I, with continuous derivatives, a and b two real numbers in I. Then :

$$\int_a^b u(t)v'(t)dt = [u(t)\,v(t)]_a^b - \int_a^b u'(t)v(t)dt$$

Proof : uv is a differentiable function and $(uv)' = u'v + uv'$.

So $u'v = (uv)' - uv'$. As $u'v, uv'$, and $(uv)'$ are continuous we have :

$$\int_a^b u(t)v'(t)dt = \int_a^b (uv)'(t) - u'(t)v(t)\, dt$$

Using the linearity of the integral :

$$\int_a^b u(t)v'(t)dt = \int_a^b (uv)'(t)dt - \int_a^b u'(t)v(t)dt$$

And uv is an antiderivative of $(uv)'$ so :

$$\int_a^b u(t)v'(t)dt = [u(t)\,v(t)]_a^b - \int_a^b u'(t)v(t)dt$$

Example :

$\int_0^1 t\,e^t\, dt$ has the form $\int_a^b u(t)v'(t)dt$ with $u(t) = t$ and $v'(t) = e^t$, the functions u, v, u', v' being continuous and $u'(t) = 1, v(t) = e^t$.

Then $\int_0^1 t\,e^t\, dt = [te^t]_0^1 - \int_0^1 e^t dt = e - 0 - [e^t]_0^1 = e - (e-1) = 1.$

Ex : Evaluate the following integrals using integration by parts :

(a) $\int_1^e x\ln x\ dx$

(b) $\int_1^{e^3} \ln x\ dx$

(c) $\int_0^1 x\sqrt{x+1}\, dx$

(d) $\int_1^e \frac{\ln x}{x^2} dx$

(e) $\int_1^{\frac{\pi}{4}} e^x \cos x\, dx$. (2IBP)